Oxford Cambridge and RSA

Monday 05 October 2020 – Afternoon

A Level Further Mathematics B (MEI)

Y420/01 Core Pure

Time allowed: 2 hours 40 minutes

You must have:
- the Printed Answer Booklet
- the Formulae Booklet for Further Mathematics B (MEI)
- a scientific or graphical calculator

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INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **144**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **8** pages.

ADVICE
- Read each question carefully before you start your answer.

Section A (36 marks)

Answer **all** the questions.

1 Using standard summation of series formulae, determine the sum of the first n terms of the series

$$(1\times2\times4)+(2\times3\times5)+(3\times4\times6)+\dots,$$

where n is a positive integer. Give your answer in fully factorised form. **[6]**

2 **(a)** The matrices $\mathbf{M}=\begin{pmatrix}0 & 1 & a\\ 1 & b & 0\end{pmatrix}$ and $\mathbf{N}=\begin{pmatrix}b & -5\\ -1 & c\\ -1 & 1\end{pmatrix}$ are such that $\mathbf{MN}=\mathbf{I}$.

Find a, b and c. **[5]**

(b) State with a reason whether or not $\mathbf{N}$ is the inverse of $\mathbf{M}$. **[1]**

3 **In this question you must show detailed reasoning.**

Find $\displaystyle\int_0^{\frac{1}{3}} \frac{1}{\sqrt{4-9x^2}}\,\mathrm{d}x$, expressing your answer in terms of π. **[4]**

4 The roots of the equation $2x^3-5x+7=0$ are α, β and γ.

(a) Find $\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma}$. **[4]**

(b) Find an equation with integer coefficients whose roots are $2\alpha-1$, $2\beta-1$ and $2\gamma-1$. **[4]**

5 Fig. 5 shows the curve with polar equation $r = a(3+2\cos\theta)$ for $-\pi \leqslant \theta \leqslant \pi$, where a is a constant.

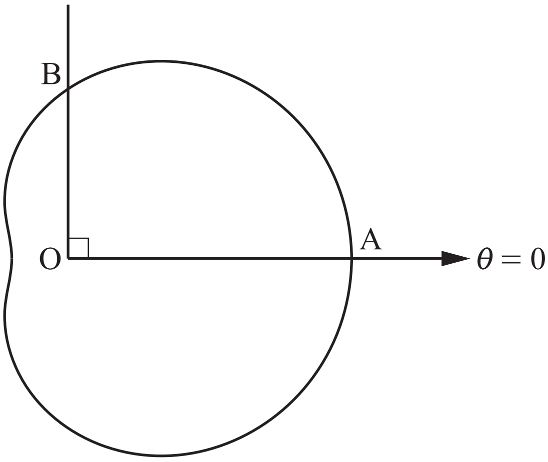

Fig. 5

(a) Write down the polar coordinates of the points A and B. **[2]**

(b) Explain why the curve is symmetrical about the initial line. **[2]**

(c) **In this question you must show detailed reasoning.**

Find in terms of a the exact area of the region enclosed by the curve. **[4]**

6 The complex number z satisfies the equation $z^2 - 4iz^* + 11 = 0$.

Given that $\mathrm{Re}(z) > 0$, find z in the form $a+b\mathrm{i}$, where a and b are real numbers. **[4]**

Section B (108 marks)

Answer **all** the questions.

7 Prove by mathematical induction that $\sum_{r=1}^{n}(r \times r!) = (n+1)! - 1$ for all positive integers n. **[6]**

8 **(a)** Given that the lines $\mathbf{r} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -1 \\ 2 \\ k \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ meet, determine k. **[5]**

(b) **In this question you must show detailed reasoning.**

Find the acute angle between the two lines. **[4]**

9 A linear transformation of the plane is represented by the matrix $\mathbf{M} = \begin{pmatrix} 1 & -2 \\ \lambda & 3 \end{pmatrix}$, where λ is a constant.

(a) Find the set of values of λ for which the linear transformation has no invariant lines through the origin. **[5]**

(b) Given that the transformation multiplies areas by 5 and reverses orientation, find the invariant lines. **[3]**

10 **In this question you must show detailed reasoning.**

The region in the first quadrant bounded by curve $y = \cosh \frac{1}{2}x^2$, the y-axis, and the line $y = 2$ is rotated through 360° about the y-axis.

Find the exact volume of revolution generated, expressing your answer in a form involving a logarithm. **[7]**

11 In this question you must show detailed reasoning.

In Fig. 11, the points A, B, C, D, E and F represent the complex sixth roots of 64 on an Argand diagram. The midpoints of AB, BC, CD, DE, EF and FA are G, H, I, J, K and L respectively.

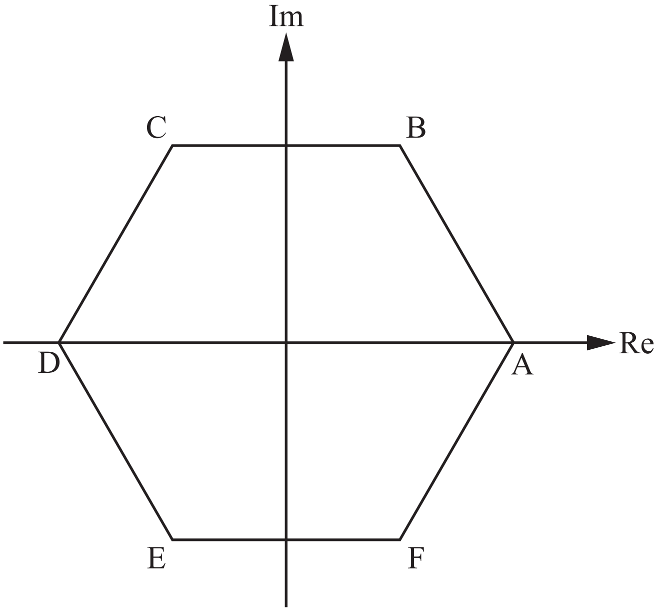

Fig. 11

(a) Write down, in exponential ($re^{i\theta}$) form, the complex numbers represented by the points A, B, C, D, E and F. **[2]**

(b) When these complex numbers are multiplied by the complex number w, the resulting complex numbers are represented by the points G, H, I, J, K and L.

Find w in exponential form. **[4]**

(c) You are given that G, H, I, J, K and L represent roots of the equation $z^6 = p$.

Find p. **[2]**

12 (a) Given that $z = \cos\theta + i\sin\theta$, express $z^n + \frac{1}{z^n}$ and $z^n - \frac{1}{z^n}$ in simplified trigonometric form. **[2]**

(b) By considering $\left(z+\frac{1}{z}\right)^3\left(z-\frac{1}{z}\right)^3$, find constants A and B such that

$$\sin^3\theta\cos^3\theta = A\sin 6\theta + B\sin 2\theta.$$ **[6]**

13 **(a)** Using exponentials, prove that $\sinh 2x = 2\cosh x \sinh x$. **[2]**

(b) Hence show that if $f(x) = \sinh^2 x$, then $f''(x) = 2\cosh 2x$. **[2]**

(c) Explain why the coefficients of odd powers in the Maclaurin series for $\sinh^2 x$ are all zero. **[2]**

(d) Find the coefficient of x^n in this series when n is a positive even number. **[3]**

14 Solve the simultaneous differential equations

$$\frac{dx}{dt} + 2x = 4y, \qquad \frac{dy}{dt} + 3x = 5y,$$

given that when $t = 0$, $x = 0$ and $y = 1$. **[11]**

15 **(a)** Show that the three planes with equations

$$\begin{aligned} x + \lambda y + 3z &= -12 \\ 2x + y + 5z &= -11 \\ x - 2y + 2z &= -9 \end{aligned}$$

where λ is a constant, meet at a unique point except for one value of λ which is to be determined. **[3]**

(b) In the case $\lambda = -2$, use matrices to find the point of intersection P of the planes, showing your method clearly. **[3]**

The line l has equation $\dfrac{x-1}{2} = \dfrac{y-1}{-1} = \dfrac{z+2}{-2}$.

(c) Find a vector equation of l. **[2]**

(d) Find the shortest distance between the point P and l. **[4]**

(e) **(i)** Show that l is parallel to the plane $x - 2y + 2z = -9$. **[3]**

(ii) Find the distance between l and the plane $x - 2y + 2z = -9$. **[2]**

16 The population density P, in suitable units, of a certain bacterium at time t hours is to be modelled by a differential equation. Initially, the population density is zero, and its long-term value is A.

(a) One simple model is to assume that the rate of change of population density is directly proportional to $A - P$.

(i) Formulate a differential equation for this model. **[1]**

(ii) Verify that $P = A(1 - e^{-kt})$, where k is a positive constant, satisfies

- this differential equation,
- the initial condition,
- the long-term condition. **[3]**

An alternative model uses the differential equation

$$\frac{dP}{dt} - \frac{P}{t(1+t^2)} = Q(t),$$

where $Q(t)$ is a function of t.

(b) Find the integrating factor for this differential equation, showing that it can be written in the form $\frac{\sqrt{1+t^2}}{t}$. **[8]**

(c) Suppose that $Q(t) = 0$.

(i) Show that $P = \frac{At}{\sqrt{1+t^2}}$. **[4]**

(ii) Find the time predicted by this model for the population density to reach half its long-term value. Give your answer correct to the nearest minute. **[2]**

(d) Now suppose that $Q(t) = \frac{te^{-t}}{\sqrt{1+t^2}}$.

Show that $P = \frac{At - te^{-t}}{\sqrt{1+t^2}}$. [You may assume that $\lim_{t\to\infty} te^{-t} = 0$.] **[5]**

It is found that the long-term value of P is 10, and P reaches half this value after 37 minutes.

(e) Determine which of the models proposed in parts **(c)** and **(d)** is more consistent with these data. **[2]**

END OF QUESTION PAPER